Convex mixed-integer optimization with Frank-Wolfe methods

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Related Papers

- **Hendrych**, Troppens, Besançon, Pokutta: Convex Integer Optimization with Frank-Wolfe Methods, *Mathematical Programming Computation*, 2025
- **Hendrych**, Besançon, Pokutta: Solving the Optimal Experiment Design Problem with Mixed-Integer Convex Methods, *Proceedings of the Symposium on Experimental Algorithms*, 2024
- Xiao, **Hendrych**, Besançon, Pokutta: Boscia.jl: A Review and Tutorial, *arXiv preprint*, 2025

Mathieu Besançon

Inria, Université Grenoble Alpes

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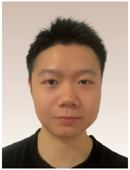

Wenjie Xiao

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Outline

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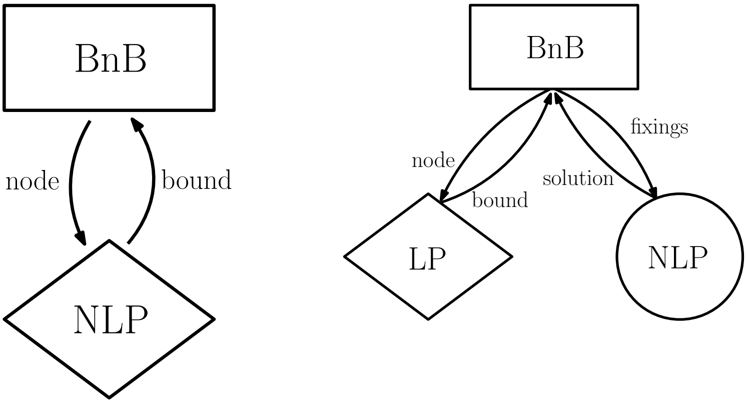
Mixed-Integer Convex Optimization

$$\begin{aligned} &\min f(\mathbf{x}) \\ &\text{s.t. } \mathbf{x} \in \mathcal{P} \\ &\quad x_j \in \mathbb{Z} \quad \forall j \in J \end{aligned}$$

where

- f is a convex, L-smooth function.
- $\mathcal{P} \subset \mathbb{R}^n$ is polytope.
- $J \subset [n]$ is a set of indices.

Common Solution Approaches



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Frank-Wolfe

Frank and Wolfe 1956; Levitin and Polyak 1966; Braun et al. 2022

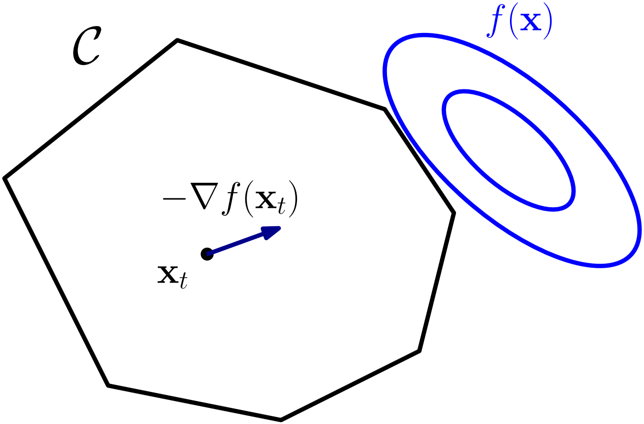
$$\min f(\mathbf{x})$$
$$\text{s.t. } \mathbf{x} \in \mathcal{X}$$

where

- f is a convex, L-smooth function.
- Oracle access to f and ∇f.
- $\mathcal{X} \subset \mathbb{R}^n$ is a compact convex set.
- $\mathcal{X}$ admits an *efficient* Linear Minimization Oracle (LMO):

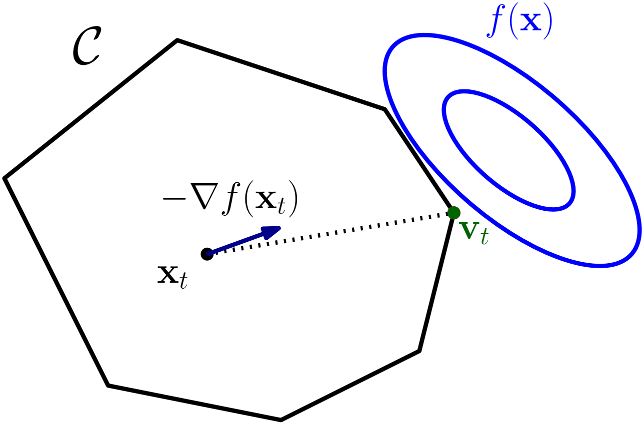
$$\arg\min_{\mathbf{v} \in \mathcal{X}} \langle \mathbf{d}, \mathbf{v} \rangle$$

Frank-Wolfe



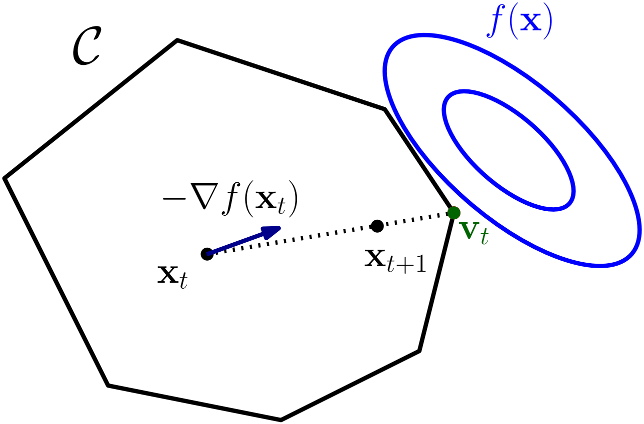
$\mathbf{x}_t$

Frank-Wolfe



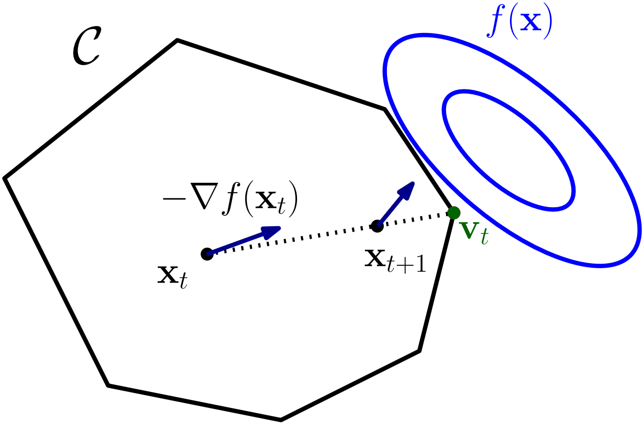
$\mathbf{x}_t$

Frank-Wolfe



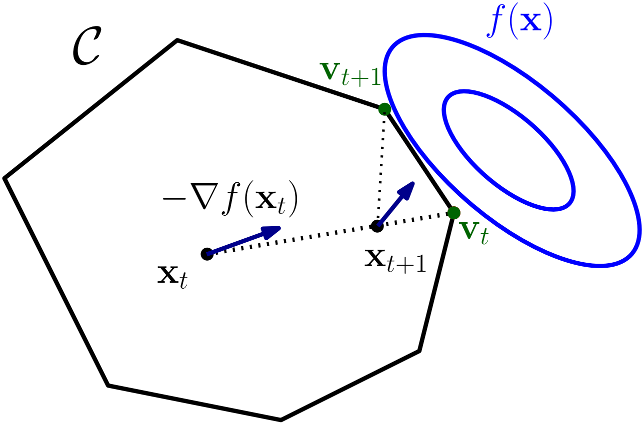
$\mathbf{x}_t$

Frank-Wolfe



$\mathbf{x}_t$

Frank-Wolfe



$\mathbf{x}_t$

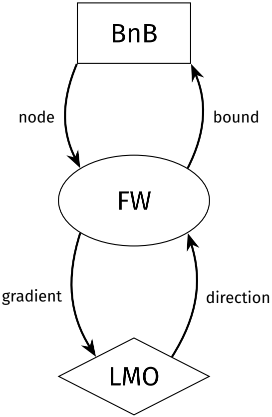
- $\mathbf{x}$ is always feasible.
- iterate $\mathbf{x}_t$ can be written as a convex combination of the vertices of $\mathcal{X}$, often stored as *active set*.
- Let $\mathbf{x}^*$ an optimal solution. Then,

$$f(\mathbf{x}) - f(\mathbf{x}^*) \leq \langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{x}^* \rangle \leq \max_{\mathbf{v} \in \mathcal{X}} \langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{v} \rangle$$

  is called the *Frank-Wolfe gap*.

BnB with Frank-Wolfe: Boscia

Hendrych et al. 2025



$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{P} \\ & x_j \in \mathbb{Z} \quad \forall j \in J \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \operatorname{conv}(\mathcal{P} \cap \mathbb{Z}_J) \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{v}} \quad & \langle \nabla f(\mathbf{x}), \mathbf{v} \rangle \\ \text{s.t.} \quad & \mathbf{v} \in \mathcal{P} \cap \mathbb{Z}_J \end{aligned}$$

Branch-and-Bound with Frank-Wolfe: `Boscia.jl`

Key Features:

- Precision-adaptive Frank-Wolfe allows for a trade-off between accuracy and computational cost.
- Warm-starting nodes via splitting the active set of the parent node.
- Aggressive reuse of information: Lazification & shadow set techniques
- Built-in heuristic: Integer feasible solutions from root node.
- No closed form of the objective function required.
- No comprehensive description of the feasible region required.

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Portfolio Optimization I

Buchheim et al. 2018

$$
\begin{aligned}
\min\ & \frac{1}{2}\Omega\, \mathbf{x}^\top M \mathbf{x} - \mathbf{r}^\top \mathbf{x} \\
\text{s.t. } & \mathbf{a}^\top \mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq 0 \\
& x_i \in \mathbb{Z} \quad \forall i \in I \subseteq [n]
\end{aligned}
$$

where M is a psd matrix.

Portfolio Optimization II

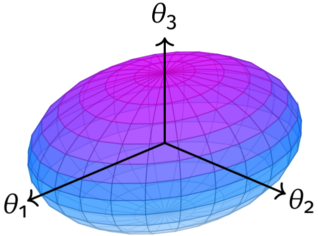
Buchheim et al. 2018

```
model = Model(HiGHS.Optimizer)
set_silent(model)
@variable(model, x[1:n], Int)
@constraint(model, x[1:n] ≥ 0)
@constraint(model, dot(a, x) ≤ b)
@constraint(model, sum(x) ≥ 1.0)
lmo = FrankWolfe.MathOptLMO(model.moi_backend)

function f(x)
    return 1 / 2 * Ω * dot(x, M, x) - dot(r, x)
end
function grad!(storage, x)
    mul!(storage, M, x, Ω, 0)
    storage .-= r
    return storage
end

x, _, result = Boscia.solve(f, grad!, lmo)
```

D-Optimal Experiment Design I

Ponte et al. 2025; Ahipaşaoğlu 2021; Li et al. 2024; Pukelsheim 2006

$$\begin{aligned} \min_{\mathbf{x}} \ & -\log\det\left(A^\top \operatorname{diag}(\mathbf{x}) A\right) \\ \text{s.t.} \ & \sum_{i=1}^{m} x_i = N \\ & \mathbf{x} \geq 0 \\ & \mathbf{x} \in \mathbb{Z}^m. \end{aligned} \tag{D}$$



D-Optimal Experiment Design II

Ponte et al. 2025; Ahipaşaoğlu 2021; Li et al. 2024; Pukelsheim 2006

```
simplex_lmo = Boscia.ProbabilitySimplexLMO(N)
lmo = Boscia.ManagedLMO(simplex_lmo, fill(0.0, m),
    fill(Float64(N), m), collect(1:m), m)

...

settings = Boscia.create_default_settings()
settings.branch_and_bound[:verbose] = true
settings.domain[:active_set] = copy(active_set)
settings.domain[:domain_oracle] = domain_oracle
settings.domain[:find_domain_point] = domain_point
settings.domain[:depth_domain] = 10
settings.frank_wolfe[:line_search] =
    FrankWolfe.Secant(domain_oracle=domain_oracle)

x, _, _ = Boscia.solve(f, grad!, lmo, settings=settings)
```

Non-Convex Quadratic Problem I

Mexi et al. 2025

$$\begin{aligned} \min\ & \mathbf{x}^\top Q\mathbf{x} + \mathbf{p}^\top \mathbf{x} + r \\ \text{s.t. } & D\mathbf{x} \leq \mathbf{c} \\ & \mathbf{x} \in \{0, 1\}^n \end{aligned}$$

- Q is symmetric but not psd.
- Tackled them heuristically during the Land-Doig Competition.
- Frank-Wolfe only converges locally → no valid lower bound.
- Wish: Plug-in user-provided lower bound.

Non-Convex Quadratic Problem II

Mexi et al. 2025

```
model = Model(HiGHS.Optimizer)
@variable(model, x[1:n], Bin)
@constraint(model, x[1:n] ≥ 0)
@constraint(model, D*x ≤ c)
lmo = FrankWolfe.MathOptLMO(model.moi_backend)

function f(x)
  return 1 / 2 * dot(x, Q, x) + dot(q, x)
end
function grad!(storage, x)
  mul!(storage, Q, x, 1, 0)
  storage .-= q
  return storage
end

settings = Boscia.create_default_settings(mode=Boscia.HEURISTIC_
    MODE)
settings.branch_and_bound[:verbose] = true
x, _, result = Boscia.solve(f, grad!, lmo, settings=settings)
```

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Outlook

Practicality and UI

- Connect to `MathOptInterface` (MOI) and `JuMP`.
- Web service.
- Precompiled and executable version.

Outlook

Practicality and UI

- Connect to `MathOptInterface` (MOI) and `JuMP`.
- Web service.
- Precompiled and executable version.

Capabilities

- Preprocessing and propagation.
- Convergence with extended smoothness definitions.
- Non-convex objective function.

Thank you for your attention!

Convex mixed-integer optimization with Frank–Wolfe methods

Boscia.jl

References I

Ahipaşaoğlu, Selin Damla (2021). "A branch-and-bound algorithm for the exact optimal experimental design problem". In: *Statistics and Computing* 31.5, p. 65.

Braun, Gábor, Alejandro Carderera, Cyrille W Combettes, Hamed Hassani, Amin Karbasi, Aryan Mokhtari, and Sebastian Pokutta (2022). "Conditional gradient methods". In: *arXiv preprint arXiv:2211.14103*.

Buchheim, Christoph, Marianna De Santis, Francesco Rinaldi, and Long Trieu (2018). "A Frank–Wolfe based branch-and-bound algorithm for mean-risk optimization". In: *Journal of Global Optimization* 70.3, pp. 625–644.

Frank, Marguerite and Philip Wolfe (1956). "An algorithm for quadratic programming". In: *Naval research logistics quarterly* 3.1-2, pp. 95–110.

Hendrych, Deborah, Hannah Troppens, Mathieu Besançon, and Sebastian Pokutta (June 28, 2025). "Convex Integer Optimization with Frank-Wolfe Methods". In: *Mathematical Programming Computation*. DOI: 10.1007/s12532-025-00288-w. arXiv: 2208.11010 [math.OC]. URL: https://link.springer.com/article/10.1007/s12532-025-00288-w.

Levitin, Evgeny S and Boris T Polyak (1966). "Constrained minimization methods". In: *USSR Computational mathematics and mathematical physics* 6.5, pp. 1–50.

Li, Yongchun, Marcia Fampa, Jon Lee, Feng Qiu, Weijun Xie, and Rui Yao (2024). "D-Optimal Data Fusion:: Exact and Approximation Algorithms". In: *INFORMS Journal on Computing* 36.1, pp. 97–120.

References II

Mexi, Gioni, Deborah Hendrych, Sébastien Designolle, Mathieu Besançon, and Sebastian Pokutta (Aug. 2, 2025). *A Frank-Wolfe-based Primal Heuristic for Quadratic Mixed-integer Optimization*. arXiv: 2508.01299 [math.OC].

Ponte, Gabriel, Marcia Fampa, and Jon Lee (2025). "Branch-and-bound for D-Optimality with fast local search and variable-bound tightening". In: *Mathematical Programming*, pp. 1–38.

Pukelsheim, Friedrich (2006). *Optimal design of experiments*. SIAM.